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Local Polynomial Estimation for SPD with Applications to VaR*

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Abstract: The local polynomial fitting method is applied to estimate the state-price density (SPD) implicit in financial asset prices. It is shown that the estimator of SPD is consistent under mild conditions. Then the VaR (Value at Risk) based on SPD is proposed, which incorporates economic valuations, and thus is more reasonable than the traditional VaR. Simulations and test of invalidation are done for Black-Scholes models to assess the performance of the proposed method.

Keywords: state-price density; local polynomial estimation; value at risk

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1 Introduction

Risk management has become a first-order concern of financial managers. The fundamental problem in the risk management is how to measure financial risks. For example, the well-known Markowitz's portfolio theory is based on the risk measure characterized by variances of risky assets. Many risk measures in finance have been proposed, among which are variances or standard deviations, semi-variances, volatility coefficients, market risk β, δ coefficients, VaR etc. See Jorion[1] and Penza and Bansal[2] for details. It is worth mentioning that VaR now becomes very popular in the risk management. Jorion[1] in a milestone book on VaR, defines VaR this way: "VaR summaries the expected maximum loss (or worst loss) over a target horizon within a given confidence level." In Aït-Sahalia and Lo's words[3], "VaR is an estimate, with predefined confidence interval, of how much one can lose from holding a position over a set horizon." Indeed, VaR has become one of the most important measure in the financial risk metrics.

Although measures such as variances and VaR do capture the important features of uncertainty, they are measures purely in statistical senses and completely ignore the economic valuation. To overcome the drawbacks of traditional financial risk measures, Aït-Sahalia and Lo[3] proposed a new measure, called E-VaR. Based upon the state-price density implicit in financial asset prices, the E-VaR inherits many good properties of the traditional statistical VaR and at the same time incorporates economic valuation. The reason is that the SPD has two characteristics. One is that it can be derived either in preference-based equilibrium models or in the arbitrage-based models of Black-Scholes[4] and Merton[5]. The other is that it covers

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all economically pertinent information, such as investors' preferences, endowments, asset price dynamics, and market clearing etc[3].

It is crucial to estimate the SPD appropriately by the asset prices in order to calculate the E-VaR. There were many discussions about estimation of the SPD. Hutchinson *et al*[6] gave the estimator of the SPD by estimating option prices nonparametrically. Rubinstein[7] proposed the estimator of the SPD by means of implied binomial tree, in which the risk-neutral probabilities $\{\pi_i^*\}$ are estimated by minimizing the sum of squared deviations between $\{\pi_i^*\}$ and a set of prior risk-neutral probabilities. Aït-Sahalia and Lo[3,8] applied the nonparametric and the semi-parametric methods to estimate the SPD. They observed that the SPD is the second derivative of the price of a call option, and thus differentiated an option pricing formula twice to obtain an estimator of the SPD.

In this paper, we apply the local polynomial method to directly estimate the SPD. As Fan and Gijbels[9] pointed out, the local polynomial fitting is superior to the other commonly used kernel estimators such as the Nadaraya-Watson (N-W) estimator and Gasser-Müller (G-M) estimator at least in three aspects. Firstly, the N-W estimator leads to undesirable form of the bias, while the G-M estimator has to pay a price in variance when dealing with a random design model. Secondly, the local polynomial method adapts to various types of designs, i.e., random and fixed designs. Finally, the method has no boundary effects: the bias at the boundary stays at the same order as that in the interior, without use of specific boundary kernels. Furthermore, what makes the local polynomial method appealing is that it applies the least squares principle and the regression function, thus the derivatives of the unknown function can be estimated at the same time. Here we employ the local polynomials to directly estimate the second derivative of the option prices and then give the estimator of the SPD. The performance of the estimator of the SPD is also discussed. Aït-Sahalia and Duarte[10] also discussed the estimation of the SPD by the local polynomial approach, but they did not give proofs of the asymptotic properties of the estimators.

The remainder of this article is arranged as follows. The backgrounds and definition of the SPD are introduced in Section 2. In Section 3, we give the consistent estimator of the SPD by the local polynomial approach. In Section 4, E-VaR is defined and compared with S-VaR. Simulations and test of invalidation are done in Section 5. All proofs are put in Appendix.

2 The state-price densities

Implicit in financial asset prices is the state-price density. It gives the price of a security that pays one dollar if a given state of nature arises, and zero otherwise. With the suitable assumptions for preferences and endowments, under the condition of market completeness which ensure that a representative investor with utility function U exists, Aït-Sahalia and Lo[3] show that the date-t price of a security with a single date-T liquidating payoff $\psi(w_T)$ is given by

$$\begin{aligned} P_t &= E_t\big[\psi(w_T)M_{t,T}\big] = \int_0^\infty \psi(w_T)\frac{U'(w_T)}{U'(w_t)}f_t(w_T)\mathrm{d}w_T \\ &= \mathrm{e}^{-r(T-t)}\int_0^\infty \psi(w_T)f_t^*(w_T)\mathrm{d}w_T \triangleq \mathrm{e}^{-r(T-t)}E_t^*\psi(w_T), \qquad (1) \end{aligned}$$

where $f_t^*(w_T) = e^{r(T-t)} M_{t,T} f_t(w_T)$ is called the state-price density (SPD), $M_{t,T} = \frac{U'(w_T)}{U'(w_t)}$ the marginal rate of substitution, and $f_t(\cdot)$ the density of the underlying S_T on which the security is written, with $dS_t = \mu(S_t, t)S_t dt + \sigma(S_t, t)S_t dZ_t$.

The statistical VaR (S-VaR) is based on the price density $f_t(\cdot)$. So it does not reflect economic valuations. But the SPD $f_t^*(\cdot)$ covers all economically pertinent information, such as investors' preferences, endowments, asset price dynamics and market clearing. Therefore, the VaR based on the SPD (E-VaR) is more appealing than S-VaR. In other words, it is more reasonable to use the SPD $f_t^*(\cdot)$ to measure financial risk than the price density $f_t(\cdot)$. The SPD $f_t^*(\cdot)$ can be applied in the same way as the price density $f_t(\cdot)$ in the risk management, and it has intimate relations with the price density $f_t(\cdot)$. Note that f_t is related to the data generating process $\{S_t\}$, whereas f_t^* aggregates all economically pertinent information. Aït-Sahalia and Lo[8] summarized their relation as follows: any two of the following imply the third: 1) representative agent's preferences; 2) asset-price dynamics; 3) the SPD.

3 Local polynomial estimator of the SPD

It follows from (1) that the date-t price P_t of a security with a single liquidating date-T payoff $\psi(S_T)$ is given by

$$P_t = e^{-r_{t,\tau}\tau} E_t^* \psi(S_T) = e^{-r_{t,\tau}\tau} \int_0^\infty \psi(S_T) f_t^*(S_T) dS_T, \tag{2}$$

where $\tau = T - t$. A European call option, written on a stock with date-t price S_t, strike price X and maturity time T, has payoff $\psi(S_T) = \max(S_T - X, 0)$. Under the assumptions of Black-Scholes[4] and Merton[5], according to (2), the date-t price H of the European call option is as follows

$$\begin{aligned} H(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau}) &= e^{-r_{t,\tau}\tau} \int_0^\infty \max(S_T - X, 0) f_t^*(S_T) dS_T \\ &= e^{-r_{t,\tau}\tau} \int_X^\infty (S_T - X) f_t^*(S_T) ds_T, \end{aligned} \tag{3}$$

where $r_{t,\tau}$ is the risk-free interests rate between t and T, and $\delta_{t,\tau}$ the dividend yield. It is easily seen from (3) that $\frac{\partial^2 H}{\partial X^2} = e^{-r_{t,\tau}\tau} f_t^*(X)$. Then $f_t^*(X) = e^{r_{t,\tau}\tau} \frac{\partial^2 H}{\partial X^2}$. This indicates that the SPD $f_t^*(\cdot)$ is the second partial derivative of the price H of a European call option scaled by the compound interests rate. In order to reduce the dimension of the explanatory variables on which the option price H depends, Aït-Sahalia and Lo[8] made the assumption that the SPD depends on S_t, $r_{t,\tau}$ and $\delta_{t,\tau}$ through the forward price $F_{t,\tau} = S_t e^{(r_{t,\tau} - \delta_{t,\tau})\tau}$ and the risk-free rate $r_{t,\tau}$. Thus the number of the independent variables of H reduces to four, i.e., $H = H(F_{t,\tau}, X, \tau, r_{t,\tau})$.

To estimate $f_t^*(\cdot)$, we only need to estimate $\frac{\partial^2 H}{\partial X^2}$. We follow Fan and Gijbels' idea about local polynomial modeling. Let $Z = (F_{t,\tau}, X, \tau, r_{t,\tau})'$, and $Z_0 = (F_{t_0,\tau_0}, X_0, \tau, r_{t_0,\tau_0})'$. If $H = H(F_{t,\tau}, X, \tau, r_{t,\tau})$ is smooth and Z is in the neighborhood of Z_0, then H can be approximated

by

$$
\begin{aligned}
H(F_{t,\tau},X,\tau,r_{t,\tau}) \simeq\ & H(Z_0)+\frac{\partial H}{\partial F_{t,\tau}}\Big|_{Z=Z_0}(F_{t,\tau}-F_{t_0,\tau_0})+\frac{\partial H}{\partial X}\Big|_{Z=Z_0}(X-X_0)\\
&+\frac{\partial H}{\partial \tau}\Big|_{Z=Z_0}(\tau-\tau_0)+\frac{\partial H}{\partial r_{t,\tau}}\Big|_{Z=Z_0}(r_{t,\tau}-r_{t_0,\tau_0})\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial F_{t,\tau}^2}\Big|_{Z=Z_0}(F_{t,\tau}-F_{t_0,\tau_0})^2\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial X^2}\Big|_{Z=Z_0}(X-X_0)^2+\frac{1}{2}\frac{\partial^2 H}{\partial \tau^2}\Big|_{Z=Z_0}(\tau-\tau_0)^2\\
&+\frac{1}{2}\frac{\partial^2 H}{\partial r_{t,\tau}^2}\Big|_{Z=Z_0}(r_{t,\tau}-r_{t_0,\tau_0})^2+\frac{\partial^2 H}{\partial F_{t,\tau}\partial X}\Big|_{Z=Z_0}(F_{t,\tau}-F_{t_0,\tau_0})(X-X_0)\\
&+\frac{\partial^2 H}{\partial F_{t,\tau}\partial \tau}\Big|_{Z=Z_0}(F_{t,\tau}-F_{t_0,\tau_0})(\tau-\tau_0)\\
&+\frac{\partial^2 H}{\partial F_{t,\tau}\partial r_{t,\tau}}\Big|_{Z=Z_0}(F_{t,\tau}-F_{t_0,\tau_0})(r_{t,\tau}-r_{t_0,\tau_0})\\
&+\frac{\partial^2 H}{\partial X\partial \tau}\Big|_{Z=Z_0}(X-X_0)(\tau-\tau_0)+\frac{\partial^2 H}{\partial X\partial r_{t,\tau}}\Big|_{Z=Z_0}(X-X_0)(r_{t,\tau}-r_{t_0,\tau_0})\\
&+\frac{\partial^2 H}{\partial \tau\partial r_{t,\tau}}\Big|_{Z=Z_0}(\tau-\tau_0)(r_{t,\tau}-r_{t_0,\tau_0}). \qquad (4)
\end{aligned}
$$

This suggests that a locally weighted polynomial regression can be employed. Let

$$
\begin{aligned}
Q(\beta) =\ & \sum_{i=1}^{n}\{P_{t_i}-\beta_0-\beta_1(F_{t_i,\tau_i}-F_{t,\tau})-\beta_2(X_i-X)\\
&-\beta_3(\tau_i-\tau)-\beta_4(r_{t_i,\tau_i}-r_{t,\tau})-\beta_5(F_{t_i,\tau_i}-F_{t,\tau})^2\\
&-\beta_6(X_i-X)^2-\beta_7(\tau_i-\tau)^2-\beta_8(r_{t_i,\tau_i}-r_{t,\tau})^2\\
&-\beta_9(F_{t_i,\tau_i}-F_{t,\tau})(X_i-X)-\beta_{10}(F_{t_i,\tau_i}-F_{t,\tau})(\tau_i-\tau)\\
&-\beta_{11}(F_{t_i,\tau_i}-F_{t,\tau})(r_{t_i,\tau_i}-r_{t,\tau})-\beta_{12}(X_i-X)(\tau_i-\tau)\\
&-\beta_{13}(X_i-X)(r_{t_i,\tau_i}-r_{t,\tau})-\beta_{14}(\tau_i-\tau)(r_{t_i,\tau_i}-r_{t,\tau})\}^2K_h(Z_i-Z), \qquad (5)
\end{aligned}
$$

where $\{P_{t_i}\}_1^n$ and $\{Z_i=(F_{t_i,\tau_i},X_i,\tau_i,r_{t_i,\tau_i})'\}_1^n$, are realizations of the option price H and its explanatory variables $Z=(F_{t,\tau},X,\tau,r)'$, $\beta=(\beta_0,\beta_1,\cdots,\beta_{14})'$ are parameters, and $K_h(\cdot)$ is a four-variable kernel function with bandwidth h.

Denote by $\hat\beta_j\ (j=0,1,\cdots,14)$, the minimizer of (5). Obviously, an estimator of $\frac{\partial^2 H}{\partial X^2}$ is $2\hat\beta_6$. Therefore an estimator of the SPD $f_t^*(\cdot)$ is given by $\hat f_t^*=2e^{r_{t,\tau}\tau}\hat\beta_6$. For convenience of

notations, let

$$X = \begin{pmatrix} 1 & F_{t_1,\tau_1} - F_{t,\tau} & \cdots & r_{t_1,\tau_1} - r_{t,\tau} & (F_{t_1,\tau_1} - F_{t,\tau})^2 & \cdots & (\tau_1 - \tau)(r_{t_1,\tau_1} - r_{t,\tau}) \\ 1 & F_{t_2,\tau_2} - F_{t,\tau} & \cdots & r_{t_2,\tau_2} - r_{t,\tau} & (F_{t_2,\tau_2} - F_{t,\tau})^2 & \cdots & (\tau_2 - \tau)(r_{t_2,\tau_2} - r_{t,\tau}) \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 1 & F_{t_n,\tau_n} - F_{t,\tau} & \cdots & r_{t_n,\tau_n} - r_{t,\tau} & (F_{t_n,\tau_n} - F_{t,\tau})^2 & \cdots & (\tau_n - \tau)(r_{t_n,\tau_n} - r_{t,\tau}) \end{pmatrix},$$

$$W = \mathrm{diag}\big(K_h(Z_1 - Z), K_h(Z_2 - Z), \cdots, K_h(Z_n - Z)\big), \quad P = (P_{t_1}, P_{t_2}, \cdots, P_{t_n})'.$$

Then (5) can be re-written as

$$Q(\beta) = (P - X\beta)'W(P - X\beta). \tag{6}$$

Minimizing (6) with respect to β, the estimator $\hat\beta$ of β is given by

$$\hat\beta = (X'WX)^{-1}X'WP. \tag{7}$$

The following theorem gives the consistency of $\hat\beta(\cdot)$ and $\hat f_t^*(\cdot)$.

Theorem 3.1 Under conditions (A1)-(A4) in Appendix, it admits that

$$\hat\beta(Z) = \beta(Z) + O\Big(\Big(\frac{\ln n}{nh_n^4}\Big)^{1/2}\Big) + O(h_n^3), \quad \text{a.s.} \tag{8}$$

$$\hat f_t^*(X) = f_t^*(X) + O\Big(\Big(\frac{\ln n}{nh_n^8}\Big)^{1/2}\Big) + O(h_n), \quad \text{a.s.} \tag{9}$$

hold uniformly for $Z \in D$ and $X \in D_1$ where $D \in \mathbf{R}^4$ and $D_1 \in \mathbf{R}^1$ are compact.

4 Economic value at risk

It follows from Section 3 that when both $\mu(S_t, t)$ and $\sigma(S_t, t)$ are constants, i.e., $\mu(S_t, t) = \mu$, $\sigma(S_t, t) = \sigma$, the price process S_t of the underlying asset satisfies

$$S_t = S_0 \exp\Big\{\Big(\mu - \frac{1}{2}\sigma^2\Big)t + \sigma Z_t\Big\}. \tag{10}$$

If we assume that $\{Z_t\}$ is a Weiner process, then S_t is a log-normal process, i.e.,

$$S_t \sim \frac{S_0}{\sqrt{2\pi t}\sigma S_t} \exp\Big\{ - \frac{[\log(S_t/S_0) - (\mu - \frac{1}{2}\sigma^2)t]^2}{2\sigma^2 t}\Big\}. \tag{11}$$

We know from Aït-Sahalia and Lo[3] that under the hypothesis of Black-Scholes[4] and Merton[5], the date-t price H of a call option maturing at date $T = t + \tau$, with strike price X, written on a stock with date-t price S_t and dividend yield $\delta_{t,\tau}$ is as follows

$$\begin{aligned} H(S_t, X, \tau, r_{t,\tau}, \delta_{t,\tau}) &= \mathrm{e}^{-r_{t,\tau}\tau} \int_0^\infty \max(S_T, 0) f_t^*(S_T)\mathrm{d}S_T \\ &= S_t\Phi(d_1) - \mathrm{e}^{-r_{t,\tau}\tau} X\Phi(d_2), \end{aligned} \tag{12}$$

where

$$d_1 = \frac{\log(S_T/X) + (r_{t,\tau} - \delta_{t,\tau} + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau},$$

with $r_{t,\tau}$ being a risk-free interest rate between t and T, and Φ being a standard normal distribution function. It is shown by Aït-Sahalia and Lo[3] and Ziegler[11] that the SPD corresponding to the price process $\{S_t\}$ is also a log-normal process, that is

$$\begin{aligned} f_t^*(S_T) &= \mathrm{e}^{r_{t,\tau}\tau} \frac{\partial^2 H}{\partial X^2}\Big|_{X=S_T} \\ &= \frac{1}{\sqrt{2\pi\tau}\sigma S_T} \exp\Big\{ -\frac{[\log(S_T/S_t) - (r_{t,\tau} - \delta_{t,\tau} - \frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau} \Big\}, \end{aligned} \tag{13}$$

while the p.d.f. of S_T conditioned on S_t is

$$Z_T|Z_t \sim f_t(S_T) = \frac{1}{\sqrt{2\pi\tau}\sigma S_T} \exp\Big\{ -\frac{[\log(S_T/S_t) - (\mu - \frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau} \Big\}. \tag{14}$$

Let U be a utility function of a representative investor and $a = (\mu - r + \delta)/\sigma$. We define U as follows

$$a = 1: \quad U(S_T) = \log(S_T), \qquad a \neq 1: \quad U(S_T) = S_T^{1-a}/(1-a). \tag{15}$$

It is easily derived that if a representative investor has utility function (15), then his coefficient of relative risk aversion is a constant, i.e., $\rho(S_T) = a$. Especially, for $a = 0$, he is a risk neutral representative investor; for $a > 0$, he is a risk aversion representative investor. In this case, the SPD f_t^* and the corresponding p.d.f. f_t are given respectively by (13) and (14).

With a confidence level α, VaR based on f_t is given by

$$P\big(\log(S_T) - \log(S_t) > \mathrm{VaR} \mid S_t\big) = 1 - \alpha. \tag{16}$$

(16) tells us that the loss of the representative investor is no more than VaR with probability $1-\alpha$. For example, if $\alpha = 5\%$, it is about 5 times in 100 trials for the representative investor to lose more than VaR. Note that

$$\begin{aligned} P\big(\log(S_T) - \log(S_t) > \mathrm{VaR} \mid S_t\big) &= P\big(S_T > S_t \exp(\mathrm{VaR}) \mid S_t\big) \\ &= \int_{S_t \exp(\mathrm{VaR})}^{\infty} f_t(S_T)\mathrm{d}S_T. \end{aligned}$$

Then VaR can be easily determined by (16). Since $\log(S_T/S_t)$ is distributed as $N((\mu - \frac{1}{2}\sigma^2)\tau, \sigma^2\tau)$, VaR is the α-quantile of distribution $N((\mu - \frac{1}{2}\sigma^2)\tau, \sigma^2\tau)$

$$\int_{\mathrm{VaR}}^{\infty} \frac{1}{\sqrt{2\pi\tau}\sigma} \exp\Big\{ -\frac{[x - (\mu - \frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau} \Big\}\mathrm{d}x = 1 - \alpha. \tag{17}$$

This VaR only depends on the price distribution f_t of the asset and contains no information about the representative investor's risk aversion. In order for VaR to contain both statistical and economic valuations, we use the SPD f_t^* instead of the p.d.f. f_t in (16). Then we obtain the economic VaR—E-VaR as follows

$$P^*\big(\log(S_T) - \log(S_t) > \mathrm{VaR}^* \mid S_t\big) = 1 - \alpha, \tag{18}$$

where the probability is with respect to the SPD f_t^*. VaR* is called the economic Value at risk.

It follows from (13) and (18) that VaR* satisfies

$$\int_{\mathrm{VaR}^*}^{\infty} \frac{1}{\sqrt{2\pi\tau}\sigma} \exp\Big\{ -\frac{[x-(r-\delta-\frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau} \Big\} = 1-\alpha. \tag{19}$$

It is easily shown from (13) and (18) that

$$\mathrm{VaR}^* = \mathrm{VaR} + (r-\delta) - \mu. \tag{20}$$

As we know that S-VaR is only an attempt to provide a single number summarizing the total risk in a portfolio of financial assets. It is obvious from (20) that E-VaR has more components or information than S-VaR. Moreover, if $\mu = r-\delta$, which implies $a = 0$ in (15), then VaR=VaR*. Thus the representative investor is risk neutral. If $\mu > r-\delta$, which means $a > 0$ in (15) and the representative investor is risk aversion. It yields that E-VaR is smaller than S-VaR. Hence, what the investor are really thinking about the risk with respect to E-VaR is much "larger" than that to S-VaR. The difference between VaR* and VaR is $\mu-(r-\delta)$, which depends on the investor's utility function through a. It indicates from (20) that E-VaR incorporates economic informations including the representative investor's preferences, endowment, market liquidity.

5 Simulations and conclusion

To assess the performance of the local polynomial estimator of the SPD, we carried out a number of simulations on the Black and Scholes models

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}Z_t, \tag{21}$$

where μ and σ are both constants.

Similarly to that in Aït-Sahalia and Lo[8], we take $\mu = 7.95\%$, $\sigma = 10.28\%$, $r = 3\%$, $T = 252$, which means there are 252 trading days in a year, and $\tau = 120, 130, \cdots, 160$, respectively. We take initial price $S_0 = 1$ and $\{Z_t\}$ Weiner process. Then from (21), the asset price sequence $\{S_1, S_2, \cdots, S_T\}$ are generated. Based on $\{s_t\}_1^T$, the option price sequence $\{P_{t_i}\}_1^T$ are given according to formula (12).

In model (4), when τ and r are fixed constants, the option price H depends only on the future price $F_{t,\tau}$ and the strike price X. So there are only six parameters β_0, β_1, β_2, β_5, β_6, β_9 in (5), which greatly reduces computations and also alleviates the curse of dimensionality. Here the kernel function is taken as $K_h(x,y) = k_h(x)k_h(y)$ where $K_h(\cdot) = \frac{1}{h}K(\cdot/h)$ is one-variable Gaussian kernel function

$$K(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}x^2 \Big).$$

with $h = 0.0198$.

Based on the local polynomial estimator, we computed the E-VaR with confidence level $\alpha = 0.05$. We did 500 simulations in order to make the estimator robust. The result is listed in Table 1. It can be seen that the difference between the estimated E-VaR and the true E-VaR (Black-Scholes E-VaR) is less than that between E-VaR and S-VaR. Moreover, we calculated the invalidation ratio based on (18) from the asset price sequence $\{S_1, S_2, \cdots, S_T\}$. They are

roughly around $\alpha = 0.05$ according to different τ. All of these results indicate that the local polynomial estimator of SPD behaves quite good.

Table 1: Output of simulations for Black and Scholes model

Days to expiration τ	120	130	140	150	160
S-VaR	-0.0765	-0.1244	-0.1379	-0.1971	-0.1031
Estimated E-VaR	-0.1118	-0.1245	-0.1260	-0.1734	-0.1432
B-S E-VaR	-0.1006	0.1498	-0.1655	-0.2258	-0.1342
Invalidation ratio	4.55%	4.10%	8.92%	6.67%	5.43%

Appendix We make the following assumptions about the model.

Condition A1

(a) The probability density function f of Z_0 is bounded, bounded away from zero and uniformly continuous on $\mathbf{R}^4$.

(b) The probability density function $f_n(u, v)$ of $(Z_0, Z_n)'$ is bounded for all $n \geq 1$.

Condition A2

(a) The kernel K is bounded with compact support.

(b) $|G_j(u) - G_j(v)| \leq C\|u - v\|$ for all $0 \leq j \leq 5$ where $G_j(u) = u^j K(u)$ and $u^j = (u_1^{j_1}, u_2^{j_2}, u_3^{j_3}, u_4^{j_4})$ with $j = j_1 + j_2 + j_3 + j_4$, $j_l \geq 0$.

Condition A3

The prices $H = H(Z)$ has bounded and uniformly continuous third partial derivatives on $\mathbf{R}^4$ and these derivatives are Lipschitz continuous.

Condition A4

(a) $E|P_1|^\nu < \infty$ for some $\nu > 2$.

(b) The conditional density $f_{Z_0|P_1}(u|v)$ of Z_0 given P_1 exists and is bounded.

(c) The conditional density $f_{(Z_0,Z_l)|(P_1,P_{1+l})}(u|v)$ of (Z_0, Z_l) given (P_1, P_{1+l}) exists and is bounded for all $l \geq 1$.

(d) The process $\{P_i, Z_i\}$ are strongly mixing with the mixing coefficients $\alpha(k)$ satisfying

$$\sum_{j=1}^{\infty} j^a \{\alpha(j)\}^{1-2/\gamma} < \infty,$$

for $2 < \gamma < \nu$ and $a > 1 - 2/\gamma$, while

$$\alpha(k) = \sup\left\{ |P(AB) - P(A)P(B)| : A \in F_t, B \in F^{t+k} \right\},$$

here $F_t = F\{(P_i, Z_i), 1 \leq i \leq t\}$, $F^{t+k} = F\{(P_i, Z_i), t + k \leq i\}$.

(e) The mixing coefficients $\alpha(k)$ satisfy $\sum_{n=1}^{\infty} \phi_1(n) < \infty$, here

$$\phi_1(n) = \frac{nL_1(n)}{r_1(n)} \Big(\frac{nT_2}{h_n^4 \ln n} \Big)^{1/4} \alpha(r_1(n)), \quad L_1(n) = \Big(\frac{nT_n^2}{h_n^6 \ln n} \Big)^2,$$

$$r_1(n) = \frac{(nh_n^4/\ln n)^{1/2}}{T_n}, \quad T_n = \big(n \ln n (\ln \ln n)^{1+\delta}\big)^{1/2}.$$

Finally, the bandwidth h_n satisfies $h_n \to 0$ as $(n \to \infty)$ and

$$\frac{n^{1+2/\nu} h_n^4}{\ln n[(\ln n)(\ln \ln n)^{1+\delta}]^{2/\nu}} \longrightarrow \infty, \quad \text{as} \quad n \to \infty \quad \text{for some} \quad 0 < \delta < 1.$$

The proof of Theorem 3.1 Let

$$k = (k_1, k_2, k_3, k_4), \quad k! = k_1! \times k_2! \times k_3! \times k_4!,$$

$$|k| = k_1 + k_2 + k_3 + k_4, \quad Z^k = Z_1^{k_1} Z_2^{k_2} Z_3^{k_3} Z_4^{k_4},$$

$$\sum_{0 \le k \le 2} = \sum_{j=0}^{2} \sum_{k_1=0}^{j} \sum_{k_2=0}^{j} \sum_{k_3=0}^{j} \sum_{k_4=0}^{j}, \quad k_1 + k_2 + k_3 + k_4 = j,$$

$$(D^k h)(Z) = \frac{\partial^{|k|} h(Z)}{\partial {Z_1}^{k_1} \partial {Z_2}^{k_2} \partial {Z_3}^{k_3} \partial {Z_4}^{k_4}}.$$

Then (4) and (5) can be expressed, respectively, as

$$H(Z) \approx \sum_{0 \le k \le 2} \frac{1}{k!} D^k H(Z)|_{Z=Z_0} (Z - Z_0)^k, \tag{22}$$

$$Q(\beta) = \sum_{i=1}^{n} \Big(P_{t_i} - \sum_{0 \le k \le 2} b_k(Z)(Z_i - Z)^k \Big)^2 K\Big(\frac{Z_i - Z}{h} \Big). \tag{23}$$

It can be seen from (23) that $\hat{\beta}$ satisfies the following equation

$$t_{n,j}(Z) = \sum_{0 \le |k| \le 2} h^{|k|} \hat{b}_k(Z) S_{n,j+k}(Z), \quad 0 \le |j| \le 2, \tag{24}$$

where

$$t_{n,j}(Z) = \frac{1}{n} \sum_{i=1}^{n} P_{t_i} \Big(\frac{Z_i - Z}{h} \Big)^j K_h(Z_i - Z), \tag{25}$$

$$S_{n,j} = \frac{1}{n} \sum_{i=1}^{n} \Big(\frac{Z_i - Z}{h} \Big)^j K_h(Z_i - Z), \tag{26}$$

and $K_h(u) = \frac{1}{h^4} K(\frac{u}{h})$. Let $\tau_n = (\tau_{n,0}', \tau_{n,1}', \tau_{n,2}')'_{15\times 1}$, where $\tau_{n,0} = t_{n,0}(Z)$. The first element of $\tau_{n,1}$ is the value of $t_{n,j}(Z)$ when $j = (0,0,0,1)$, the second element of $\tau_{n,1}$ is the value of $t_{n,j}(Z)$ when $j = (0,0,1,0)$, and so on; the first element of $\tau_{n,2}$ is the value of $t_{n,j}(Z)$ when $j = (0,0,0,2)$, the second element of $\tau_{n,2}$ is the value of $t_{n,j}(Z)$ when $j = (0,0,1,1)$, and so on. The last element of $\tau_{n,2}$ (the tenth element) is the value of $t_{n,j}(Z)$ when $j = (2,0,0,0)$. Note that $\tau_{n,0}$, $\tau_{n,1}$, $\tau_{n,2}$ are counted with respect to $|j| = 0,1,2$, and $t_{n,j}$ are alphabetically arranged according to each j. Similarly, $h^{|k|}\hat{b}_k$, $0 \le k \le 2$ can be written as an array $\hat{\beta}_n = (\hat{\beta}_{n,0}, \hat{\beta}_{n,1}, \hat{\beta}_{n,2})'$. Note that $\hat{\beta}_n$ and $\hat{\beta}$ have the same elements except the order. Moreover, all possible values of $S_{n,j+k}$ can be written alphabetically as matrixes $S_{n,|j|,|k|}$. Let

$$S_n = \begin{pmatrix} S_{n,0,0} & S_{n,0,1} & S_{n,0,2} \\ S_{n,1,0} & S_{n,1,1} & S_{n,1,2} \\ S_{n,2,0} & S_{n,2,1} & S_{n,2,2} \end{pmatrix}_{15\times 15}.$$

Then the equation (24) can be expressed as

$$\tau_n(Z) = S_n(Z)\hat{\beta}_n(Z). \tag{27}$$

It is easily shown that S_n is positive-definite. Thus

$$\hat{\beta}_n(Z) = S_n^{-1}(Z)\tau_n(Z). \tag{28}$$

Let

$$t^*_{n,j}(Z) = \frac{1}{n}\sum_{i=1}^{n}[P_{t_i} - H(Z_i)]\Big(\frac{Z_i - Z}{h}\Big)^j K_h(Z_i - Z). \tag{29}$$

Then

$$\begin{aligned} t_{n,j}(Z) - t^*_{n,j}(Z) &= \frac{1}{n}\sum_{i=1}^{n} H(Z_i)\Big(\frac{Z_i - Z}{h}\Big)^j K_h(Z_i - Z) \\ &= \sum_{0\le|k|\le 2}\frac{1}{k!}h^{|k|}D^k H(Z)S_{n,j+k}(Z) + e_{n,j}(Z), \end{aligned}$$

where

$$\begin{aligned} e_{n,j}(Z) &= 3\sum_{|k|=3}\frac{h^3}{k!}\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{Z_i - Z}{h}\Big)^{k+j} K_h(Z_i - Z) \\ &\quad\times\int_0^1\big[(D^k H)(Z + w(Z_i - Z)) - (D^k H)(Z)\big](1-w)^2\mathrm{d}w. \end{aligned}$$

Note $D^k H = k! b_k$, then the above results and (24) yield

$$\begin{aligned} t^*_{n,j}(Z) &= \sum_{0\le|k|\le 2} h^{|k|}\big[\hat{b}_k(Z) - b_k(Z)\big]S_{n,j+k}(Z) \\ &\quad - h^3\sum_{|k|=3}\frac{1}{k!}D^k H(Z)S_{n,j+k}(Z) - e_{n,j}(Z). \end{aligned}$$

Thus, similarly to that for τ_n, $t^*_{n,j}(Z)$ can be alphabetically written as following, denoted as $\tau^*_n(Z)$

$$\tau^*_n(Z) = S_n(Z)(\hat{\beta} - \beta)(Z) - h^3 B_n(Z)H_3(Z) - e_n(Z), \tag{30}$$

where $B_n(Z) = (S_{n,0,3}, S'_{n,1,3}, S'_{n,2,3})'$, while $H_3(Z)$ and e_n are the matrix and the array, which alphabetically written with respect to $\frac{1}{j!}(D^{|j|}H)(Z)$, $|j| = 3$ and $e_{n,j}$, $0 \le |j| \le 2$. Following (30), we have

$$\hat{\beta}_n(Z) - \beta(Z) = S_n^{-1}(Z)\tau^*_n(Z) + h^3 S_n^{-1}(Z)B_n(Z)H_3(Z) + S_n^{-1}(Z)e_n(Z).$$

When all above conditions satisfied, it yields from Masry[12] that

$$\sup_{Z\in D}\big|S_{n,j}(Z) - f(Z)\mu_j\big| = O\Big(\Big(\frac{\ln n}{nh_n^4}\Big)^{1/2}\Big) + O(h_n^{\theta}), \quad \text{a.s.}$$

where $\mu_j = \int_{\mathbf{R}^4} u^j K(u)\mathrm{d}u$ and

$$\sup_{Z\in D} |e_{n,j}(Z)| = o(h_n^3), \quad \text{a.s.}$$

It is shown that (8) holds, and hence (9) holds too.

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状态价格密度的局部多项式估计及其在VaR中的应用

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摘 要: 本文应用局部多项式拟合方法来估计金融资产价格中隐含的状态价格密度(SPD)，在较弱的条件下证明了SPD的估计是相合的。然后基于SPD我们提出了包含经济价值的VaR。从这个意义上讲它比传统的VaR更合理。最后针对Black-Scholes模型进行了数值模拟和失败率检验以评价本文方法的好坏。

关键词: 状态价格密度；局部多项式估计；在险价值